Equilibrium in Nash Differential Games via Lyapunov–type iterations

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Abstract

This paper discusses the numerical solution of the coupled algebraic Riccati equations associated with the linear quadratic differential games. The Lyapunov iteration for solving the considered coupled equations is discussed by Li and Gajic in 1994. We modify this iteration and derive the new algorithm with typically convergence properties for methods of such a type introduced in the literature. Finally, in order to demonstrate the efficiency of the proposed algorithms, computational examples are provided and numerical effectiveness of the considered algorithms is commented.

JEL codes: C6, C7, C73, C020

Key Words: Linear quadratic differential game, Nash strategies, Coupled algebraic Riccati equations.

1. Introduction

Many situations in economics and management are characterized by multiple decision players and enduring consequences of decisions. The theory which conceptualizes problems of this kind is dynamic games. Dynamic game theory tries to arrive at appropriate models describing the process. The solutions of the coupled algebraic Riccati equations produce the answers to some important problems of modern control theory, for example, differential games with conflict of interest and simultaneous decision making (Nash strategies), optimal control problems, and jump linear systems. The linear quadratic control model is one of the most widely used control models in both empirical and theoretical economic modelling. Examples in this direction can be found in (McGratten, 1994), (Amman and Kendrick, 1995), (Amman and Neudecker, 1997) and many more. Examples of dynamic games in economics and management science can be found in (Dockner, Jorgensen, Long and Sorger 2000, Jorgensen and Zaccour 2003, Plasman, Engwerda, Aarle, Bartolomeo and Michalak 2006).

The solution of the coupled algebraic Riccati equations corresponding to steady state Nash strategies of the linear quadratic differential game problem is presented in terms of the Lyapunov iterations considered by Li and Gajic (Li and Gajic, 1994). The obtained solution is (stabilizing one), positive semidefinite (definite) and valid under the stabilizability-detectability assumptions imposed on the problem matrices. We consider a few iteration formulas for finding this solution. These methods are derived by applying the results and iterations of stochastic linear quadratic control problems (Freiling and Hochhaus 2004, Ivanov 2007). The notations used in this paper are fairly standard. We note with $X > 0$ ($X \geq 0$) the symmetric positive definite (semidefinite) matrix. The superscript T denotes the matrix transpose. $I$ denotes the $m \times m$ identity matrix. $|| . ||_2$ denotes the Euclidean norm for a matrix.

A controlled linear dynamic system corresponding to the Nash differential game strategies is given by

$$
\dot{x} = A_0 x + B_1 u_1 + B_2 u_2, \quad x(t_0) = x_0
$$

where $x$ is a state vector, $u_1$ and $u_2$ are control inputs (for the reason of simplicity we limit our attention to two control agents), $A_0$, $B_1$ and $B_2$ are constant matrices of appropriate dimensions.

With each control agent a quadratic type functional is associated, that is

$$
J_1(u_1, u_2, x_0) = \frac{1}{2} \int_{t_0}^{t_f} (x^T Q_1 x + u_1^T R_1 u_1 + u_2^T R_2 u_2) dt.
$$

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\[ J_2(u_1, u_2, x_0) = \frac{1}{2} \int_0^\infty (x^T Q_2 x + u_1^T R_2 u_1 + u_2^T R_2 u_2) dt. \]

Weighting matrices are symmetric and \( Q_i \geq 0, R_{12} > 0, R_{22} > 0, R_{12} \geq 0, R_{22} \geq 0 \). The optimal solution in the class of the best linear feedback laws forms the so-called Nash optimal strategies \( u_1^* \) and \( u_2^* \) satisfying
\[
J_1(u_1^*, u_2^*) \leq J_1(u_1, u_2^*) \quad \text{and} \quad J_1(u_1^*, u_2^*) \leq J_1(u_1^*, u_2) .
\]

It is well known that the closed-loop Nash optimal strategy is given by
\[
(1) \quad u_i^* = -R_i^{-1} B_i^T X_i, \quad i = 1, 2
\]

where \( X_1 \) and \( X_2 \) satisfy the coupled algebraic Riccati equations
\[
(2) \quad N_1(X_1, X_2) = 0, \quad N_2(X_1, X_2) = 0,
\]

where
\[
N_1(X_1, X_2) = X_1 A_0 + A_0^T X_1 + Q_1 - X_1 S_1 X_1 - X_1 S_2 X_2 - X_2 S_2 X_1 + X_2 S_1 X_2
\]
\[
N_2(X_1, X_2) = X_2 A_0 + A_0^T X_2 + Q_2 - X_2 S_2 X_2 - X_2 S_1 X_1 - X_1 S_2 X_2 + X_1 S_1 X_1
\]

with
\[
S_i = B_i R_i^{-1} B_i^T, \quad i = 1, 2,
\]
\[
S_{12} = B_1 R_1^{-1} R_{11} R_2^{-1} B_2^T, \quad S_{21} = B_2 R_2^{-1} R_{22} R_1^{-1} B_1^T.
\]

The existence of the nonlinear optimal Nash strategies (1) attracts the attention of many researchers. The existence of Nash strategies (1) and a solution of the coupled algebraic Riccati equations (2) were studied by many authors. Li and Gajic (Li and Gajic, 1994) have described an iteration which operates only on two decoupled standard algebraic Riccati equations and performs iterations on two algebraic Lyapunov equations.

2. Iterations for finding a stabilizing solution

It is shown that the algorithm converges to then nonnegative (positive) definite stabilizing solution of (2) under the following standard control-oriented assumption.

**Assumption.** Either the triple \((A_0, B_1, \sqrt{Q_1})\) or \((A_0, B_2, \sqrt{Q_2})\) is stabilizable-detectable.

The following iteration is proposed in (Gajic and Li, 1988) for solving the coupled algebraic Riccati equations (2)
\[
(3) \quad X_1^{(i+1)}(A_0 - S_1 X_1^{(i)} - S_2 X_2^{(i)}) + (A_0 - S_1 X_1^{(i)} - S_2 X_2^{(i)})^T X_1^{(i+1)} + Q_1 + X_1^{(i)} S_1 X_1^{(i)} + X_2^{(i)} S_2 X_2^{(i)} = 0
\]
\[
(4) \quad X_2^{(i+1)}(A_0 - S_1 X_1^{(i)} - S_2 X_2^{(i)}) + (A_0 - S_1 X_1^{(i)} - S_2 X_2^{(i)})^T X_2^{(i+1)} + Q_2 + X_2^{(i)} S_2 X_2^{(i)} + X_1^{(i)} S_1 X_1^{(i)} = 0.
\]

In order to apply the Newton method we transform the coupled Riccati equations to the new nonlinear matrix equation. Therefore, we propose the Newton method to the derived equation. For this we assume \( S_1 > 0 \) and \( S_2 > 0 \), \( S_{12} \geq 0 \), \( S_{21} \geq 0 \) and define
\[
B = \text{diag}(B_1, B_2), \quad A = \text{diag}(A_0, A_0), \quad B = \text{diag}(B_1, B_2), \quad Q = \text{diag}(Q_1, Q_2),
\]
\[
\hat{S} = \text{diag}(S_{21}, S_{12}), \quad R = \text{diag}(R_{11}, R_{22}), \quad T = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad K = S T S^{-1}.
\]

The coupled Riccati equations \( N_1(X_1, X_2) = 0 \) and \( N_2(X_1, X_2) = 0 \) are equivalent to the equation \( R(X) = 0 \) where
\[
R(X) = A^T X + X A + Q + T X (K + \hat{S}) X T + (X + T X K) S (X + T X K).
\]

We have the following properties for the function \( R(X) \). We write down
\[
R(X) = (A + F_X)^T X + X (A + F_X) + Q + W_X
\]
where

\[
W_X = \begin{pmatrix}
I & Q + TX\tilde{S}XT \\
F_X & K^T XT \\
\end{pmatrix}^{-1}
\begin{pmatrix}
I & TXK \\
F_X & S^{-1} \\
\end{pmatrix},
\]

\[
\tilde{S} = KTS + \tilde{S}. \quad F_X = -S(X + TXK)^T, \quad \Phi_{Z,Y} = (F_Z - F_Y)^T S^{-1}(F_Z - F_Y).
\]

For any symmetric matrices Y and Z we derive the following property of \( R(X) \):

\[
R(Y) = (A + F_X)^T Y + Y(A + F_X) + T(Z - Y)\tilde{S}(Z - Y)T + \Phi_{Z,Y} +
\]

\[
\begin{pmatrix}
I & (Q - T\tilde{Z}\tilde{S}XT + T\tilde{Z}YT + T\tilde{S}YT + T\tilde{K}YT) \\
F_X & K^T YT \\
\end{pmatrix}^{-1},
\]

The last inequality is used as a practical stopping criterion.

Let us consider the Newton method for \( R(X) = 0 \), it is

\[
X_{i+1} = X_i - (R'(X_i))^{-1}(R(X_i)), \quad i=0,1,2,3,\ldots
\]

The last iteration leads to the following nonlinear matrix equation

\[
(A + F_X)^T X_{i+1} + X_{i+1}(A + F_X) + Q + F_X^T S^{-1} F_X T - TX\tilde{S}X_T +
\]

\[
+TX_{i+1}(KF_X + \tilde{S}X_T) + (F_X^T K^T + TX\tilde{S})X_{i+1}T = 0
\]

for a given initial matrix \( X_0 \). In the last equation we replace \( X_{i+1} \) with \( X_i \) in the expression

\[
TX_{i+1}(KF_X + \tilde{S}X_T) + (F_X^T K^T + TX\tilde{S})X_{i+1}T
\]

and we derive the new formulas for the Lyapunov iteration to solve \( R(X) = 0 \). New iteration is

\[
(A + F_X)^T X_{i+1} + X_{i+1}(A + F_X) + Q + W_{X_i} = 0
\]

which is equivalent to the next modified iteration

\[
(A + F_X)^T X_{i+1} + X_{i+1}(A + F_X) + Q + TX\tilde{S}X_T +
\]

\[
+ (F_X^T + TX\tilde{S})T S^{-1} (F_X + TSX_T) = 0.
\]

It easy to show that iterations (6) and (7) are equivalent to decoupled Lyapunov iterations (3)-(4).

All considered iterations are executed under the initial conditions. We have to choose the matrix

\[
X_0 = \text{diag}(X_1^{(0)}, X_2^{(0)}) > 0
\]

with \( A + F_{X_0} \) is asymptotically stable and \( R(X_0) \leq 0 \).

3. Numerical examples

Our aim is to study considered iterations for finding a symmetric solution of a set of Riccati equations (2). We will carry out experiments for numerical solving an algebraic Riccati equation \( R(X) = 0 \) with introduced iterations: Newton iteration (NI) (5), and two Lyapunov iterations (6) and (7). We compare these methods with results from decoupled iteration (3)-(4) (DLI).

We have to solve a nonlinear matrix equation at each step of iteration (5). This equation is equivalent to

\[
((A + F_X)^T \otimes I + I \otimes (A + F_X)^T + T \otimes Z_{X_i} + Z_{X_i}^T \otimes T^T)\vec{X} = \vec{Q},
\]

where

\[
Z_{X_i} = K F_X + \tilde{S}X_T, \quad \vec{Q} = Q + F_X^T S^{-1} F_X^T - TX\tilde{S}X_T
\]

and the symbol \( \otimes \) is the Kronecker product and the vec operation stacks the columns of a matrix into a long vector. Solutions of iterations (6) and (7) can be found in terms of the solution of an algebraic Lyapunov equation at each step. For this purpose the MATLAB procedure dlyap is applied.

Our experiments are executed in MATLAB on an 1,81GHz PENTIUM(R) Dual CPU computer. We denote tol - a small positive real number denoting the accuracy of computation: \( E_2 = \| R(X_i) \|_2 \); It - number of iterations for which the inequality \( E_h \leq tol \). The last inequality is used as a practical stopping criterion.

For all examples we take \( B_1 \) and \( B_2 \) to be \( \text{num} \) square matrices of the form
\[
B_1 = \begin{bmatrix}
0.02 & 0.02 & 0 \\
0.001 & 0.02 & \ddots \\
0 & \ddots & \ddots \\
0.001 & 0.02 & 0.001
\end{bmatrix},
B_2 = B_1^T.
\]

and the remain matrices are diagonal ones and defined as follows \( Q_1 = (0.1)I, Q_2 = (0.3)I, R_{11} = 3I, R_{22} = 4I, R_{21} = 5I, R_{21} = 6I \).

We have executed three set of examples with different values of \( m \). We compare all iterations introducing the following parameters: "\( m_{\text{It}} \)" - the biggest number of iterations; "\( \text{av It} \)" - the average number of iterations; "\( N \)" - the number of examples where the corresponding iteration does not converge in considered tolerance \( tol \). We name these examples "unsolved" ones. Probably the convergence conditions do not fulfilled for initial matrix \( X_0 = \text{diag}(X_1^{(0)}, X_2^{(0)}) \). To determine the numbers \( m_{\text{It}} \) and \( \text{av It} \) we count those examples of each size for which the corresponding iteration converges. We choose \( tol = 10^{-10} \) for all different tests.

We have constructed the three sets of examples for testing the introduced iterations. For the first test of examples the coefficient the matrix \( A_0 \) was constructed as follows:

Test 1: \( A_0 = \text{rand}(m)/5 - (0.45)I, \)

Test 2: \( A_0 = \text{rand}(m)/6 - (0.125)I, \)

Test 3: \( A_0 = \text{rand}(m)/6 - (0.15)I, \)

where \( \text{rand}(m) \) returns an \( m \)-by-\( m \) matrix of pseudorandom scalar values drawn from a uniform distribution on the unit interval (see MATLAB description). Hundred examples of \( m = 4, 5, 6, 7, 8, 9, 10 \) of three set of equations were generated and solved via the considered iterations. All tables report the maximal number of iterations \( m_{\text{It}} \) and \( \text{av It} \) of each size for all examples needed for achieving the stoping criterion. We have carried out experiments with all iterations for the initial point \( X_0 = \text{diag}(X_1^{(0)}, X_2^{(0)}), X_1^{(0)} = X_2^{(0)} = 3I \).

<table>
<thead>
<tr>
<th>( m )</th>
<th>( m_{\text{It}} )</th>
<th>( \text{av It} )</th>
<th>( N )</th>
<th>( m_{\text{It}} )</th>
<th>( \text{av It} )</th>
<th>( N )</th>
<th>( m_{\text{It}} )</th>
<th>( \text{av It} )</th>
<th>( N )</th>
<th>( m_{\text{It}} )</th>
<th>( \text{av It} )</th>
<th>( N )</th>
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<td>23</td>
<td>4.8</td>
<td>1</td>
<td>23</td>
<td>4.8</td>
<td>1</td>
<td>19</td>
<td>3.9</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>7</td>
<td>3.1</td>
<td>0</td>
<td>9</td>
<td>4.5</td>
<td>1</td>
<td>9</td>
<td>4.5</td>
<td>1</td>
<td>7</td>
<td>3.6</td>
<td>1</td>
</tr>
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<td>6</td>
<td>3.2</td>
<td>0</td>
<td>121</td>
<td>6.2</td>
<td>0</td>
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<td>0</td>
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<tr>
<td>12</td>
<td>7</td>
<td>3.4</td>
<td>0</td>
<td>19</td>
<td>6.0</td>
<td>1</td>
<td>19</td>
<td>6.0</td>
<td>1</td>
<td>16</td>
<td>4.8</td>
<td>1</td>
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<td>14</td>
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<td>9</td>
<td>3.7</td>
<td>0</td>
<td>50</td>
<td>7.5</td>
<td>2</td>
<td>50</td>
<td>7.5</td>
<td>2</td>
<td>40</td>
<td>6.3</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 1: Results from experiments for Test 1.

All examples are solved
6 examples are unsolved
6 examples are unsolved
6 examples are unsolved
Table 2: Results from experiments for Test 2.

<table>
<thead>
<tr>
<th></th>
<th>Newton (5)</th>
<th>LI1 (6)</th>
<th>LI2 (7)</th>
<th>DLI (3)-(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2m</td>
<td>mIt avIt N</td>
<td>mIt avIt N</td>
<td>mIt avIt N</td>
<td>mIt avIt N</td>
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<tr>
<td>8</td>
<td>11 3.4 0</td>
<td>13 5.8 2</td>
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<tr>
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<tr>
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<tr>
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<td>169 10.4 2</td>
<td>137 8.3 3</td>
</tr>
<tr>
<td>20</td>
<td>9 4.2 1</td>
<td>143 11.1 0</td>
<td>143 11.1 0</td>
<td>114 8.8 1</td>
</tr>
</tbody>
</table>

4 examples are unsolved, 13 examples are unsolved, 13 examples are unsolved, 16 examples are unsolved.

Table 3: Results from experiments for Test 3.

<table>
<thead>
<tr>
<th></th>
<th>Newton (5)</th>
<th>LI1 (6)</th>
<th>LI2 (7)</th>
<th>DLI (3)-(4)</th>
</tr>
</thead>
<tbody>
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<td>mIt avIt N</td>
<td>mIt avIt N</td>
<td>mIt avIt N</td>
<td>mIt avIt N</td>
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</tr>
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<td>399 12.2 1</td>
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<td>20</td>
<td>34 4.4 2</td>
<td>192 11.5 3</td>
<td>70 9.6 4</td>
<td>23 7.4 6</td>
</tr>
</tbody>
</table>

5 examples are unsolved, 15 examples are unsolved, 19 examples are unsolved, 25 examples are unsolved.

4. Conclusion

We have made numerical experiments for computing a Hermitian solution to a system of two algebraic Riccati equations with several iterative methods. An example is given by Li and Gajic (Li and Gajic, 1994) where it is demonstrated that iteration DLI (3)-(4) is very effective for solving (2). Our numerical experiments confirm that this is so in general. We have compared the results from these experiments in regard of number of iterations.

All iterations achieve the same accuracy for different number of iterations. Newton’s method has quadratic convergence, it executes the least of all number of iterations, it achieves the best accuracy and it is not the fastest because it performs huge computational work per one iteration. It is assumed for the other methods that they have the linear rate of convergence.
References


